

Massive Lossless Data Compression and Multiple Parameter Estimation from Galaxy Spectra

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We present a method for radical linear compression of datasets where the data are dependent on some number M of parameters. We show that, if the noise on the data is independent of the parameters, we can form M linear combinations of the data which contain as much information about all the parameters as the entire dataset, in the sense that the Fisher information matrices are identical; i.e. the method is lossless. We explore how these compressed numbers fare when the noise is dependent on the parameters, and show that the method, although not precisely lossless, increases errors by a very modest factor. The method is general, but we illustrate it with a problem for which it is well-suited: galaxy spectra, whose data typically consist of $\sim 10^3$ fluxes, and whose properties are set by a handful of parameters such as age, brightness and a parametrised star formation history. The spectra are reduced to a small number of data, which are connected to the physical processes entering the problem. This data compression offers the possibility of a large increase in the speed of determining physical parameters. This is an important consideration as datasets of galaxy spectra reach 10^6 in size, and the complexity of model spectra increases. In addition to this practical advantage, the compressed data may offer a classification scheme for galaxy spectra which is based rather directly on physical processes.

I. INTRODUCTION

There are many instances where objects consist of many data, whose values are determined by a small number of parameters. Often, it is only these parameters which are of interest. The aim of this paper is to find linear combinations of the data which are focussed on estimating the physical parameters with as small an error as possible. Such a problem is very general, and has been attacked in the case of parameter estimation in large-scale structure and the microwave background (e.g. [1] [hereafter TTH], [2], [3], [4]). Previous work has concentrated largely on the estimation of a single parameter; the main advance of this paper is that it sets out a method for the estimation of multiple parameters. The method provides one projection per parameter, with the consequent possibility of a massive data compression factor. Furthermore, if the noise in the data is independent of the parameters, then the method is entirely lossless. i.e. the compressed dataset contains as much information about the parameters as the full dataset, in the sense that the Fisher information matrix is the same for the compressed dataset as the entire original dataset. An equivalent statement is that the mean likelihood surface is at the peak locally identical when the full or compressed data are used.

We illustrate the method with the case of galaxy spectra, for which there are surveys underway which will provide $\sim 10^6$ objects. In this application, the noise is generally not independent of the parameters, as there is a photon shot-noise component which depends on how many photons are expected. We take a spectrum with poor signal-to-noise, whose noise is approximately from photon counting alone, and investigate how the method fares. In this case, the method is not lossless, but the increase in error bars is shown to be minimal, and superior in this respect to an alternative compression system PCA (Principal Component Analysis).

One advantage such radical compression offers is speed of analysis. One major scientific goal of galaxy spectral surveys is to determine physical parameters of the stellar component of the galaxies, such as the age, star formation history, initial mass function and so on. Such a process can, in principle, be achieved by generating model galaxy spectra by stellar population synthesis techniques, and finding the best-fitting model by maximum-likelihood techniques. This can be very time-consuming, and must inevitably be automated for so many galaxies. In addition, one may have a large parameter space to explore, so any method which can speed up this process is worth investigation. One possible further application of the data compression method is that the handful of numbers might provide the basis of a classification scheme which is based on the physical properties one wants to measure.

The outline of the paper is as follows: in section II we set out the lossless compression method for noise which is independent of the parameters; the proof appears in the appendix. In section III we discuss the more general case where the noise covariance matrix and the mean signal both depend on the parameters. In section IV we show through a worked example of galaxy spectra that the method, although not lossless, works very well in the general case.

II. METHOD

We have a set of n data, represented by a vector \mathbf{x}_i , $i = 1, \dots, n$. The data values depend on a set of parameters θ_α , and contain noise, which could be correlated. If the noise is gaussian, its statistical properties are determined entirely by the covariance matrix $\mathbf{C}_{ij} = \langle \mathbf{x}_i \mathbf{x}_j \rangle - \langle \mathbf{x}_i \rangle \langle \mathbf{x}_j \rangle$. For galaxy spectra, \mathbf{x}_i is a set of (rest-frame) flux densities, and the parameters may be, for example, age, magnitude of source, metallicity and some parameters describing the star formation history.

The aim is to derive the parameters from the data. One approach is simply to perform a maximum likelihood solution by exploring all parameter space, using all n pixels. We try to find a number $n' < n$ of (linear) combinations of the spectral data \mathbf{x} which encompass as much as possible of the information about the physical parameters. We find that this can be done losslessly in some circumstances; the spectra can be reduced to a handful of numbers without loss of information. The speed-up in parameter estimation by factors ~ 100 is useful when the galaxy and model datasets are large.

The information content is given by the Fisher information matrix,

$$\mathbf{F}_{\alpha\beta} \equiv -\left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial \theta_\alpha \partial \theta_\beta} \right\rangle \quad (1)$$

where $\mathcal{L}(\theta_\alpha)$ is the likelihood, which we will assume to be gaussian. The average is over an ensemble with the same parameters but different noise. For the full dataset,

$$\mathcal{L}(\theta_\alpha) = \frac{1}{(2\pi)^{n/2} \sqrt{\det(\mathbf{C})}} \exp \left[-\frac{1}{2} \sum_{i,j} (x_i - \mu_i) \mathbf{C}_{ij}^{-1} (x_j - \mu_j) \right] \quad (2)$$

where $\boldsymbol{\mu}(\theta_\alpha) \equiv \langle \mathbf{x} \rangle$ is the mean and in general \mathbf{C} also depends on θ_α .

For a single parameter, the Fisher matrix \mathbf{F} is just a scalar F , and the error on the parameter can be no smaller than $F^{-1/2}$. For many parameters, the error on one parameter α (marginalised over the others) is at least $[(\mathbf{F}^{-1})_{\alpha\alpha}]^{1/2}$. There is some discussion of the Fisher matrix in [1], hereafter TTH. For future reference, the Fisher matrix for the entire data set is (TTH, equation (15))

$$\mathbf{F}_{\alpha\beta} = \frac{1}{2} \text{Tr} [\mathbf{C}^{-1} \mathbf{C}_{,\alpha} \mathbf{C}^{-1} \mathbf{C}_{,\beta} + \mathbf{C}^{-1} (\boldsymbol{\mu}_{,\alpha} \boldsymbol{\mu}_{,\beta}^t + \boldsymbol{\mu}_{,\beta} \boldsymbol{\mu}_{,\alpha}^t)] \quad (3)$$

where $\boldsymbol{\mu} = \langle \mathbf{x} \rangle$, and the comma indicates derivative with respect to the parameter. This represents the best that can possibly be done with the data. The aim of the linear compression method is to find a linear combination of the original data, to produce a single number

$$y \equiv \mathbf{b}^t \mathbf{x} \quad (4)$$

for some vector \mathbf{b} (t indicates transpose), which minimises the error on a parameter θ_α . If we fix all the other parameters, this amounts to maximising the remaining diagonal component of the Fisher information matrix. After compression to y , the Fisher matrix is

$$\mathbf{F}_{\alpha\beta} = \frac{1}{2} \left(\frac{\mathbf{b}^t \mathbf{C}_{,\alpha} \mathbf{b}}{\mathbf{b}^t \mathbf{C} \mathbf{b}} \right) \left(\frac{\mathbf{b}^t \mathbf{C}_{,\beta} \mathbf{b}}{\mathbf{b}^t \mathbf{C} \mathbf{b}} \right) + \frac{(\mathbf{b}^t \boldsymbol{\mu}_{,\alpha})(\mathbf{b}^t \boldsymbol{\mu}_{,\beta})}{(\mathbf{b}^t \mathbf{C} \mathbf{b})} \quad (5)$$

Note that the denominators are simply numbers. Clearly, we can multiply \mathbf{b} by a constant without changing the information content, so we choose for convenience to maximise a diagonal component $\mathbf{F}_{\alpha\alpha}$ subject to the constraint

$$\mathbf{b}^t \mathbf{C} \mathbf{b} = 1. \quad (6)$$

In general the mean $\boldsymbol{\mu}$ and the covariance matrix \mathbf{C} both depend on the parameters of the spectrum, and the resulting maximisation leads to an eigenvalue problem which is nonlinear in \mathbf{b} . We are unable to solve this, so we consider a

case for which an analytic solution can be found. TTH showed how to solve for the case of estimation of a single parameter in two special cases: 1) when $\boldsymbol{\mu}$ is known, and 2) when \mathbf{C} is known (i.e. doesn't depend on the parameters). We will concentrate on the former case, but generalise to the problem of estimating many parameters at once. For a single parameter, TTH showed that the entire dataset could be reduced to a single number, with no loss of information about the parameter. We show below that, if we have M parameters to estimate, then we can reduce the dataset to M numbers. These M numbers contain just as much information as the original dataset; i.e. the data compression is lossless.

We consider the parameters in turn. First, we maximise \mathbf{F}_{11} , which requires

$$\frac{\partial}{\partial b_i} (2b_j \mu_{,1j} b_k \mu_{,1k} b_k - \lambda b_j C_{jk} b_k) = 0 \quad (7)$$

where λ is a Lagrange multiplier. Following TTH, this gives a single eigenvector, here normalised to comply with the constraint:

$$\mathbf{b}_1 = \frac{\mathbf{C}^{-1} \boldsymbol{\mu}_{,1}}{\sqrt{\boldsymbol{\mu}_{,1}^t \mathbf{C}^{-1} \boldsymbol{\mu}_{,1}}} \quad (8)$$

where $y_1 = \mathbf{b}_1^t \mathbf{x}$. To see whether this is lossless, we compare the Fisher matrix element before and after the compression. Substitution of \mathbf{b}_1 into (5) gives

$$\mathbf{F}_{11} = \boldsymbol{\mu}_{,1}^t \mathbf{C}^{-1} \boldsymbol{\mu}_{,1} \quad (9)$$

which is identical to the Fisher matrix element using the full data (equation 3) if \mathbf{C} is independent of θ_1 . Hence, as claimed by TTH, the compression from the *entire* dataset to the single number y_1 loses no information about θ_1 . For example, if $\boldsymbol{\mu} \propto \theta$, then $y_1 = \sum_i \mathbf{x}_i / \sum_i \boldsymbol{\mu}_i$ and is simply an estimate of the parameter itself.

A. Estimation of many parameters

The question is how can we generalise this to the case of many parameters. We find a second number $y_2 \equiv \mathbf{b}_2^t \mathbf{x}$ by the following requirements:

- The new y_2 is uncorrelated with y_1 . This demands that $\mathbf{b}_2^t \mathbf{C} \mathbf{b}_1 = 0$.
- The new number should capture as much information about the second parameter θ_2 as possible.

This requires two Lagrange multipliers (we normalise \mathbf{b}_2 by demanding that $\mathbf{b}_2^t \mathbf{C} \mathbf{b}_2 = 1$ as before). Maximising and applying the constraints gives the solution

$$\mathbf{b}_2 = \frac{\mathbf{C}^{-1} \boldsymbol{\mu}_{,2} - (\boldsymbol{\mu}_{,2}^t \mathbf{b}_1) \mathbf{b}_1}{\sqrt{\boldsymbol{\mu}_{,2}^t \mathbf{C}^{-1} \boldsymbol{\mu}_{,2} - (\boldsymbol{\mu}_{,2}^t \mathbf{b}_1)^2}}. \quad (10)$$

This is readily generalised to any number M of parameters. There are then M orthogonal vectors \mathbf{b}_m , $m = 1, \dots, M$, each y_m capturing as much information about parameter α_m which is not already contained in y_q ; $q < m$. The constrained maximisation gives

$$\mathbf{b}_m = \frac{\mathbf{C}^{-1} \boldsymbol{\mu}_{,m} - \sum_{q=1}^{m-1} (\boldsymbol{\mu}_{,m}^t \mathbf{b}_q) \mathbf{b}_q}{\sqrt{\boldsymbol{\mu}_{,m}^t \mathbf{C}^{-1} \boldsymbol{\mu}_{,m} - \sum_{q=1}^{m-1} (\boldsymbol{\mu}_{,m}^t \mathbf{b}_q)^2}}. \quad (11)$$

This procedure is analogous to Gram-Schmidt orthogonalisation with a curved metric, with \mathbf{C} playing the role of the metric tensor.

Since, by construction the numbers y_m are uncorrelated, the likelihood of the parameters is obtained by multiplication of the likelihoods obtained from each statistic y_m . In other words, the Fisher matrix of the combined numbers is just the sum of the individual Fisher matrices. The y_m have mean and variance given by

$$\langle y_m \rangle = \mathbf{b}_m^t \boldsymbol{\mu}; \quad \sigma_m^2 = \mathbf{b}_m^t \mathbf{C} \mathbf{b}_m. \quad (12)$$

1. Proof that the method can be lossless for many parameters

Under the assumption that the covariance matrix is independent of the parameters, reduction of the original data to the M numbers y_m results in no loss of information about the M parameters at all. In fact the set $\{y_m\}$ produces, on average, a likelihood surface which is locally identical to that from the entire dataset – no information about the parameters is lost in the compression process. A proof of this for an arbitrary number of parameters is given in the appendix.

III. THE GENERAL CASE

In general, the covariance matrix does depend on the parameters, and this is the case for galaxy spectra, where at least one component of the noise is parameter-dependent. This is the photon counting noise, for which $\mathbf{C}_{ii} = \boldsymbol{\mu}_i$. TTH argued that it is better to treat this case by using the n eigenvectors which arise from assuming the mean is known, rather than the single number (for one parameter) which arises if we assume that the covariance matrix is known, as above. We find that, on the contrary, the small number of eigenvectors \mathbf{b}_m allow a much greater degree of compression than the known-mean eigenvectors (which in this case are simply individual pixels, ordered by $|\boldsymbol{\mu}_{,\alpha}/\boldsymbol{\mu}|$). For data signal-to-noise of around 2, the latter allow a data compression by about a factor of 2 before the errors on the parameters increase substantially, whereas the method here allows drastic compression from thousands of numbers to a handful. To show what can be achieved, we use a set of simulated galaxy spectra to constrain a few parameters characterising the galaxy star formation history.

A. Parameter Eigenvectors

In the case when the covariance matrix is independent of the parameters, it does not matter which parameter we choose to form y_1, y_2 , etc, as the likelihood surface from the compressed numbers is, on average, locally identical to that from the full dataset. However, in the general case, the procedure does lose information, and the amount of information lost will depend on the order of assignment of parameters to m . If the parameter estimates are correlated, as we will see in Fig. 2, the error in both parameters is dominated by the length of the likelihood contours along the ‘ridge’. It makes sense then to diagonalise the matrix of second derivatives of $\ln \mathcal{L}$ at the fiducial model, and use these as the parameters (temporarily), as proposed by [5] for galaxy surveys. The parameter eigenvalues would order the importance of the parameter combinations to the likelihood. The procedure would be to take the smallest eigenvalue (with eigenvector lying along the ridge), and make the likelihood surface as narrow as possible in that direction. One then repeats along the parameter eigenvectors in increasing order of eigenvalue.

Specifically, diagonalise $\mathbf{F}_{\alpha\beta}$ in (3), to form a diagonal covariance matrix $\Lambda = \mathbf{S}^t \mathbf{F} \mathbf{S}$. The orthogonal parameter combinations are $\psi = \mathbf{S}^t \theta$, where \mathbf{S} has the normalised eigenvectors of \mathbf{F} as its columns. The weighting vectors \mathbf{b}_m are then computed from (11) by replacing $\boldsymbol{\mu}_{,\alpha p}$ by $\mathbf{S}_{pr} \boldsymbol{\mu}_{,\alpha r}$.

IV. A WORKED EXAMPLE: GALAXY SPECTRA

We start by investigating a two-parameter model. We have run a grid of stellar evolution models, with a burst of star formation at time $-t$, where $t = 0$ is the present day. The star formation rate is $SFR(t') = A\delta(t' + t)$ where δ is a Dirac delta function. The two parameters to determine are age t and normalisation A . Fig. 1 shows some spectra with fixed normalisation ($1M_\odot$ of stars produced) and different age. There are $n = 352$ pixels between 300 and 1000 nm. Real data will be more complicated (variable transmission, instrumental noise etc) but this system is sufficiently complex to test the methods in essential respects. For simplicity, we assume that the noise is gaussian, with a variance given by the mean, $\mathbf{C} = \text{diag}(\boldsymbol{\mu}_1, \dots)$. This is appropriate for photon number counts when the number is large. We assume the same behaviour, even with small numbers, for illustration, but there is no reason why a more complicated noise model cannot be treated. It should be stressed that this is a more severe test of the model than a typical galaxy spectrum, where the noise is likely to be dominated by sources independent of the galaxy, such as CCD read-out noise or sky background counts. In the latter case, the compression method will do even better than the example here. The simulated galaxy spectrum is one of the galaxy spectra (age 3.95 Gyr, model number 100), and the maximum signal-to-noise per bin is taken to be 2. Noise is added, approximately photon noise, with a gaussian distribution with variance equal to the number of photons in each channel (Fig. 1). Hence $\mathbf{C} = \text{diag}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots)$. The most probable values for the age and normalisation (assuming uniform priors) is given by maximising the likelihood:

$$\mathcal{L}(\text{age, norm}) = \frac{1}{(2\pi)^{n/2} \sqrt{\prod_i \mu_i}} \exp \left[-\frac{1}{2} \sum_i (x_i - \mu_i)^2 / \mu_i \right] \quad (13)$$

where $\boldsymbol{\mu}$ depends on age and normalisation. In \mathcal{L} is shown in Fig. 2. Since this uses all the data, and all the approximations hold, this is the best that can be done, given the S/N of the spectrum. To solve the eigenvalue problem for \mathbf{b} requires an initial guess for the spectrum. This ‘fiducial model’ was chosen to have an age of 8.98 Gyr, i.e. very different from the true solution (model number 150 rather than 100). This allows us to compute the eigenvector \mathbf{b}_1 from (8). This gives the single number $y_1 = \mathbf{b}_1^t \mathbf{x}$. With this as the datum, the likelihood for age and normalisation is

$$\mathcal{L}(\text{age, norm}) = \frac{1}{\sqrt{2\pi}\sigma_{y_1}} \exp \left[-\frac{(y_1 - \langle y_1 \rangle)^2}{2\sigma_{y_1}^2} \right] \quad (14)$$

where $\langle y_1 \rangle = \mathbf{b}_1^t \boldsymbol{\mu}$ and $\sigma_{y_1}^2 = \mathbf{b}_1^t \mathbf{C} \mathbf{b}_1$. Note that the mean and covariance matrix here depend on the parameters - i.e. they are not from the fiducial model. The resultant likelihood is shown in Fig. 4. Clearly it does less well than the full solution, but it does constrain the parameters to a narrow ridge, on which the true solution (age model=100, log(normalisation) = 0) lies. The second eigenvector \mathbf{b}_2 is obtained by taking the normalisation as the second parameter. The vector is shown in the lower panel of Fig. 3. The normalisation parameter is rather a special case, which results in \mathbf{b}_2 differing from \mathbf{b}_1 only by a constant offset in the weights (For this parameter $\boldsymbol{\mu}_{,\alpha} = \boldsymbol{\mu}$ and so $\mathbf{C}^{-1} \boldsymbol{\mu}_{,\alpha} = (1, 1, \dots, 1)^t$). The likelihood for the parameters with y_2 as the single datum is shown in Fig. 5. On its own, it does not tightly constrain the parameters, but when combined with y_1 , it does remarkably well (Fig. 6).

A. Three-parameter estimation

We complicate the situation now to a 3-parameter star-formation rate $SFR(t) = A \exp(-t/\tau)$, and estimate A , t and τ . Chemical evolution is included by using a simple closed-box model (with instantaneous recycling; [6]). This affects the depths of the absorption lines. If we follow the same procedure as before, choosing (t, A, τ) as the order for computing \mathbf{b}_1 , \mathbf{b}_2 and \mathbf{b}_3 , then the product of the likelihoods from y_1 , y_2 and y_3 is as shown in the right panel of Fig. 7. The left panel shows the likelihood from the full dataset of 1000 numbers, which does little better than the 3 compressed numbers. It is interesting to explore how the parameter eigenvector method fares in this case. Here we follow the procedure in section 2, and maximise the curvature along the ridge first. The resulting three numbers constrain the parameters as in the middle panel; in this case there is no apparent improvement over using eigenvectors from (t, A, τ) , but it may be advantageous in other applications.

B. Estimate of increase in errors

For the noise model we have adopted, we can readily compute the increase in the conditional error for one of the parameters - the normalisation of the spectrum. This serves as an illustration of how much information is lost in the compression process. In this case, $\mathbf{C} = \boldsymbol{\mu}$, and $\mathbf{C}_{,\alpha} = \boldsymbol{\mu}_{,\alpha} = \boldsymbol{\mu}$, and the Fisher matrix (a single element) can be written in terms of the total number of photons and the number of spectral pixels. From (3), $F^O = N_{\text{photons}} + N_{\text{pixels}}/2$. The compressed data, on the other hand, have a Fisher matrix $F = N_{\text{photons}} + 1/2$, so the error bar on the normalisation is increased by a factor

$$\text{Fractional error increase} \simeq \sqrt{1 + \frac{1}{2s}} \quad (15)$$

for $N_{\text{photons}} \gg 1$, and $s \equiv N_{\text{photons}}/N_{\text{pixels}}$ is the average number of photons per pixel. Even if s is as low as 2, we see that the error bar is increased only by around 12%.

C. Computational issues

We have reduced the likelihood problem in this case by a factor of more than a hundred. The eigenproblem is trivial to solve. The work to be done is in reducing a whole suite of model spectra to M numbers, by forming scalar products of them with the vectors \mathbf{b}_m , and computing the errors on the y_m , using (12). This is a one-shot task, and trivial in comparison with the job of generating the models.

D. Role of fiducial model

The fiducial model sets the weightings \mathbf{b}_m . After this step, the likelihood analysis is correct for each y_m , even if the fiducial model is wrong. The only place where there is an approximation is in the multiplication of the likelihoods from all y_m to estimate finally the parameters. The y_m are strictly only uncorrelated if the fiducial model coincides with the true model. This approximation can be dropped, if desired, by computing the correlations of the y_m for each model tested. We have explored how the fiducial model affects the recovered parameters, and an example result from the two-parameter problem is shown in Fig. 8. Here the ages and normalisations of a set of ‘true’ galaxies with $S/N \lesssim 2$ are estimated, using a common (9Gyr) galaxy as the fiducial model. We see that the method is successful at recovering the age, even if the fiducial model is very badly wrong. There are errors, of course, but the important aspect is whether the compressed data do significantly worse than the full dataset of 352 numbers. Fig. 8 shows that this is not the case.

Although it appears from this example to be unnecessary, if one wants to improve the solution, then it is permissible to iterate, using the first estimate as the fiducial model. This adds to the computational task, but not significantly; assuming the first iteration gives a reasonable parameter estimate, then one does not have to explore the entire parameter space in subsequent iterations.

V. COMPARISON WITH PRINCIPAL COMPONENT ANALYSIS

It is interesting to compare with other data compression and parameter estimation methods. Principal Component Analysis (e.g. [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]), for example, projects the data onto eigenvectors of the covariance matrix, which is determined empirically from the scatter between flux measurements of different galaxies. Part of the covariance matrix in PCA is therefore determined by differences in the models, whereas in our case \mathbf{C} refers to the noise alone. PCA then finds uncorrelated projections which contribute in decreasing amounts to the variance between galaxies in the sample.

One finds that the first principal component is correlated with the galaxy age [17]. Figure 9 shows the PCA eigenvectors obtained from a set of 20 burst model galaxies which differ only in age, and Figure 10 shows the resultant likelihood from the first two principal components. In the language of this paper, the principal components are correlated, so the 2×2 covariance matrix is used to determine the likelihood. We see that the components do not do nearly as well as the parameter eigenvectors; they do about as well as y_1 on its own. For interest, we plot the first principal component and y_1 vs. age in Figure 11. In the presence of noise ($S/N < 2$ per bin), y_1 is almost monotonic with age, whereas PC1 is not. Since PCA is not optimised for parameter estimation, it is not lossless, and it should be no surprise that it fares less well than the tailored eigenfunctions of section III.

VI. DISCUSSION

We have presented a linear data compression algorithm for estimation of multiple parameters from an arbitrary dataset. If there are M parameters, the method reduces the data to a compressed dataset with M members. In the case where the noise is independent of the parameters, the compression is lossless; i.e. the M data contain as much information about the parameters as the entire dataset. Specifically, this means the mean likelihood surface around the peak is locally identical whichever of the full or compressed dataset is used as the data. This can result in a massive compression, with the degree of compression given by the ratio of the size of the dataset to the number of parameters. Parameter estimation is speeded up by the same factor.

Although the method is lossless in certain circumstances, we believe that the data compression can still be very effective when the noise does depend on the model parameters. We have illustrated this using simulated galaxy spectra as the data, where the noise comes from photon counting (in practice, other sources of noise will also be present, and possibly dominant); we find that the algorithm is still almost lossless, with errors on the parameters increasing typically by a factor $\sim \sqrt{1 + 1/(2s)}$, where s is the average number of photons per spectral channel. The example we have chosen is a more severe test of the algorithm than real galaxy spectra; in reality the noise may well be dominated by factors external to the galaxy, such as detector read-out noise, sky background counts (for ground-based measurements) or zodiacal light counts (for space telescopes). In this case, the noise is indeed independent of the galaxy parameters, and the method is lossless.

The compression method requires prior choice of a fiducial model. The fiducial model parameters determines the projection vectors. The choice of fiducial model will not bias the solution, and the likelihood given the y_m can be computed without approximation. In our examples, we find that the method correctly recovers the true solution,

even if the fiducial model is very different. There are circumstances where the choice of a good fiducial model may be more important, if the eigenvectors depend very sensitively on the model parameters. An example of this is the determination of the redshift z of the galaxy, whose observed wavelengths are increased by a factor $1 + z$ by the expansion of the Universe. If the main signal for z comes from spectral lines, then the method will give great weight to certain discrete wavelengths, determined by the fiducial z . If the true redshift is different, these wavelengths will not coincide with the spectral lines. It should be stressed that the method will still allow an estimate of the parameters, including z , but the error bars will not be optimal. This may be one case where applying the method iteratively may be of great value.

We have compared the parameter estimation method with another linear compression algorithm, Principal Component Analysis. PCA is not lossless unless all principal components are used, and compares unfavourably with this method. Other, more ad hoc, schemes consider particular features in the spectrum, such as broad-band colours, or equivalent widths of lines [19]. Each of these is a ratio of linear projections, with weightings given by the filter response or sharp filters concentrated at the line. There may well be merit in the way the weightings are constructed, but they will not in general do as well as the optimum weightings presented here. It is worth remarking on the ability of the method to separate parameters such as age and metallicity, which often appear degenerately in some methods. In the ‘external noise’ case, then *provided* the degeneracy can be lifted by maximum likelihood methods using every pixel in the spectrum, then it can also be lifted by using the reduced data. A complication which may arise in a real galaxy spectrum is the presence of features not in the model, such as emission lines from hot gas. These can be included if the model is extended by inclusion of extra parameters. This problem exists whether the full or compressed data are used. Of course, we can use standard goodness-of-fit tests to determine whether the data are consistent with the model as specified, or whether more parameters are required.

The data compression to a handful of numbers offers the possibility of a classification scheme for galaxy spectra. This is attractive as the numbers are connected closely with the physical processes which determine the spectrum, and will be explored in a later paper. An additional realistic aim is to determine the star formation history of each individual galaxy, without making specific assumptions about the form of the star formation rate. The method in this paper provides the means to achieve this.

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Appendix

In this appendix, we prove that the linear compression algorithm for estimation of an arbitrary number M of parameters is lossless, provided the noise is independent of the parameters, $\mathbf{C}_{,\alpha} = 0$. Specifically, loss-free means the Fisher matrix for the set of M numbers $y_m = \mathbf{b}_m^t \mathbf{x}$ is identical to the Fisher matrix of the original dataset \mathbf{x} :

$$F_{\alpha\beta}^O = \langle \alpha | \beta \rangle. \quad (16)$$

By construction, the y_m are uncorrelated, so the likelihoods multiply and the Fisher matrix for the set $\{y_m\}$ is the sum of the derivatives of the log-likelihoods from the individual y_m :

$$F_{\alpha\beta} = \sum_m F_{\alpha\beta}(m). \quad (17)$$

From (5),

$$F_{\alpha\beta}(m) = (\mathbf{b}_m^t \boldsymbol{\mu}_{,\alpha})(\mathbf{b}_m^t \boldsymbol{\mu}_{,\beta}) \quad (18)$$

With (11), we can write

$$\mathbf{b}_m^t = \frac{\boldsymbol{\mu}_{,m}^t \mathbf{C}^{-1} - \sum_{q=1}^{m-1} (\mathbf{b}_q^t \boldsymbol{\mu}_{,m}^t) \mathbf{b}_q}{\sqrt{\langle m | m \rangle - \sum_{q=1}^{m-1} (\mathbf{b}_q^t \boldsymbol{\mu}_{,m}^t)^2}}. \quad (19)$$

Hence

$$F_{\alpha\beta}(m) = \left[\langle \alpha | m \rangle - \sum_{q=1}^{m-1} F_{\alpha m}(q) \right] \times$$

$$\frac{\left[\langle\beta|m\rangle - \sum_{q=1}^{m-1} F_{\beta m}(q)\right]}{\left[\langle m|m\rangle - \sum_{q=1}^{m-1} F_{mm}(q)\right]} \quad (20)$$

Consider first $\beta = m$:

$$\begin{aligned} F_{\alpha m}(m) &= \langle\alpha|m\rangle - \sum_{q=1}^{m-1} F_{\alpha m}(q) \\ \Rightarrow F_{\alpha M} &= \sum_{q=1}^M F_{\alpha M}(q) = \langle\alpha|M\rangle = F_{\alpha M}^O \end{aligned} \quad (21)$$

proving that these terms are unchanged after compression. We therefore need to consider $F_{\alpha\beta}(m)$ for α or $\beta < m$. First we note that

$$F_{\alpha\beta}(m) = \frac{F_{\alpha m}(m)F_{m\beta}(m)}{F_{mm}(m)} \quad (22)$$

and, from (21),

$$\sum_{q=1}^{\beta} F_{\alpha\beta}(q) = \langle\alpha|\beta\rangle \quad (23)$$

We want the sum to extend to M . However, the terms from $\beta + 1$ to M are all zero. This can be shown as follows: (22) shows that it is sufficient to show that $F_{\alpha m}(m) = 0$ if $m > \alpha$. Setting $\beta = m$ in (23), and reversing α and m , we get

$$\sum_{\alpha+1}^m F_{\alpha m}(q) = 0. \quad (24)$$

Now, the contribution from q does not depend on derivatives wrt higher-numbered parameters, so we can evaluate $F_{\alpha m}(\alpha + 1)$ by setting $m = \alpha + 1$. The sum (24) implies that this term is zero. Increasing m successively by one up to M , and using (24), proves that all the terms are zero, proving that the compression is lossless.

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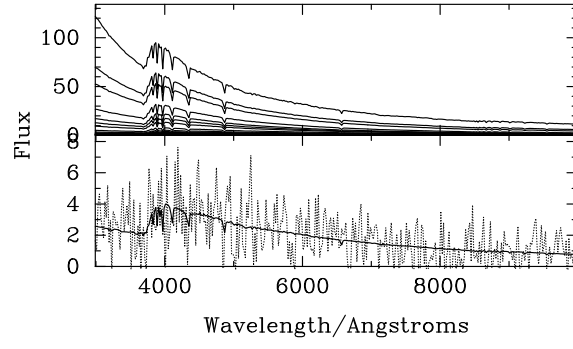


FIG. 1. Top panel: example model spectra, with age increasing downwards. Bottom panel: simulated galaxy spectrum (including noise), whose properties we wish to determine, superimposed on noise-free spectrum of galaxy with the same age.

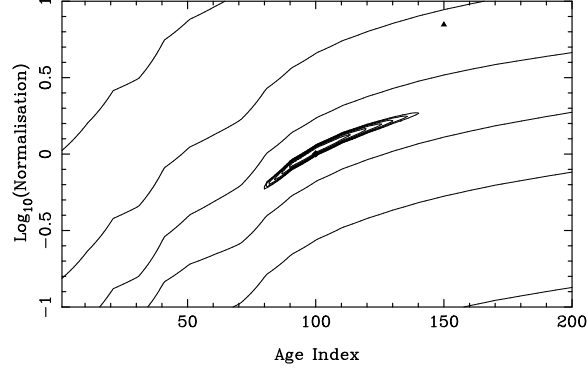


FIG. 2. Full likelihood solution using all pixels. There are 6 contours running down from the peak value in steps of 0.5 (in $\ln \mathcal{L}$), and 3 outer contours at -100 , -1000 and -10000 . The triangle in the upper-right corner marks the fiducial model which determines the eigenvectors $\mathbf{b}_{1,2}$.

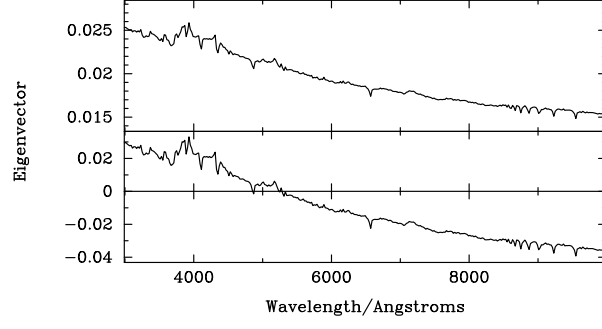


FIG. 3. Eigenvectors $-\mathbf{b}_1$ (age) and $-\mathbf{b}_2$ (normalisation). Wavelength λ is in Angstroms. Note that the weights in \mathbf{b}_1 are negative, which is why the sign has been changed for plotting: the blue (left) end of the spectrum which is weighted most heavily for y_1 . This is expected as this part of the spectrum changes most rapidly with age. Note that these weightings differ by a constant; this feature is special to the amplitude parameter, and is explained in the text.

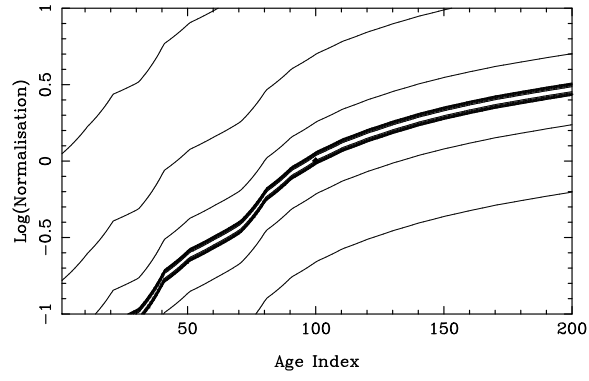


FIG. 4. Likelihood solution for the age datum y_1 . Contours are as in Fig. 2.

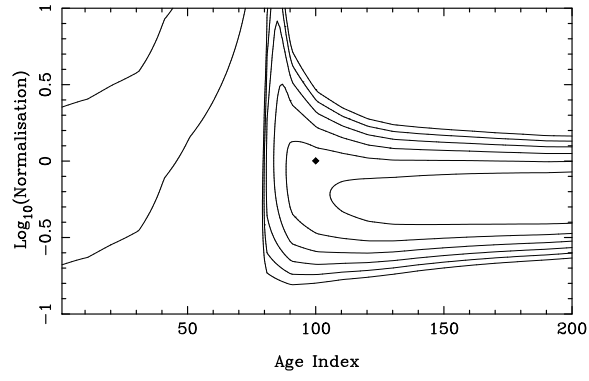


FIG. 5. Likelihood solution for the normalisation datum y_2 . Contours are as in Fig. 2.

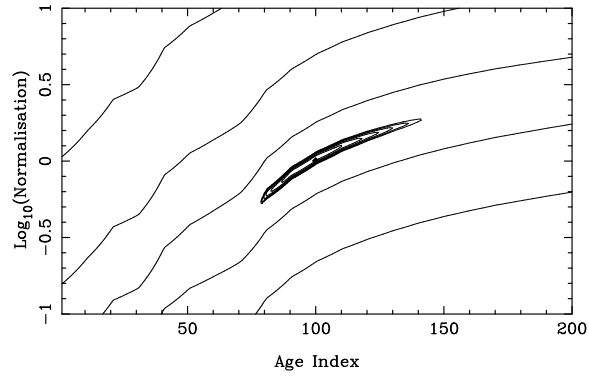


FIG. 6. Likelihood solution for the age datum y_1 and the normalisation datum y_2 . Contours are as in Fig. 2.

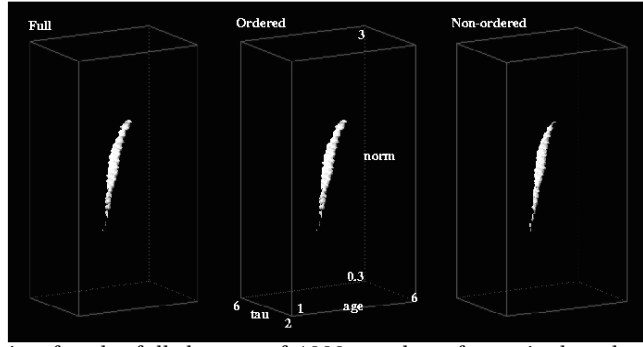


FIG. 7. (Left) Likelihood solution for the full dataset of 1000 numbers for a single galaxy, as a function of t/Gyr , τ/Gyr and amplitude. (Middle) Likelihood for 3 compressed numbers, from parameter eigenvectors. (Right) likelihood surface from 3 compressed numbers (age, normalisation and τ eigenvectors). All contours shown are 3.13 below the peak in $\ln \mathcal{L}$; the irregularities in the surface are artefacts of the surface-drawing routine.

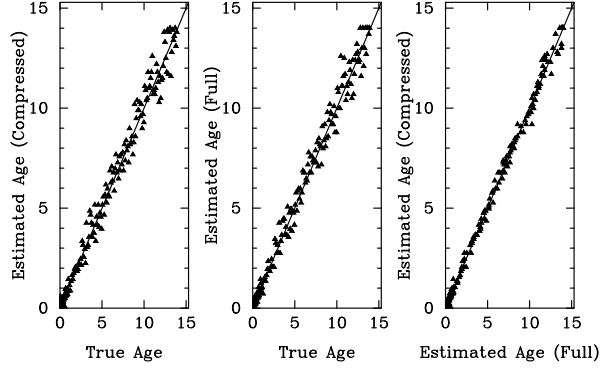


FIG. 8. The effect of the fiducial model on recovery of the parameters. Here a single fiducial model is chosen (with age 9 Gyr), and ages recovered from many true galaxy spectra with ages between zero and 14 Gyr. Top left panel shows the recovered age from the two numbers y_1 and y_2 (with age and normalisation weightings), plotted against the true model age. Top right shows how well the full dataset (with $S/N \lesssim 2$) can recover the parameters. The lower panel shows the estimated age from the y_1 and y_2 plotted against the age recovered from the full dataset, showing that the compression adds very little to the error, even if the fiducial model is very wrong. Note also that the scatter increases with age; old galaxies are more difficult to date accurately.

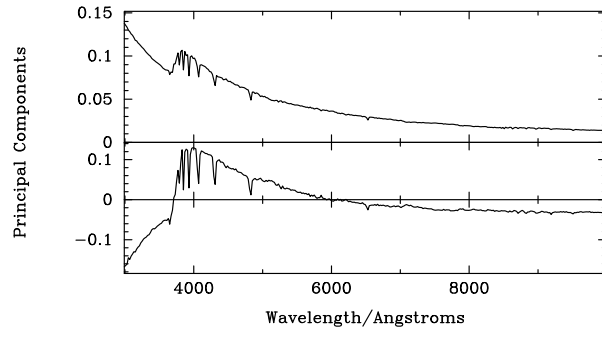


FIG. 9. The first two principal component eigenvectors, from a system of model spectra consisting of a burst at different times.

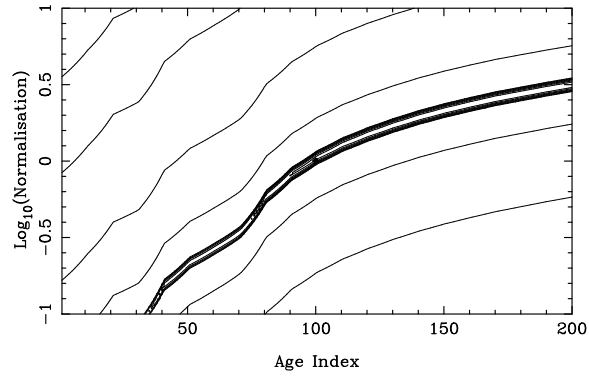


FIG. 10. Likelihood solution for the first two principal components, PC1 (top) and PC2. Contours are as in Fig. 2.

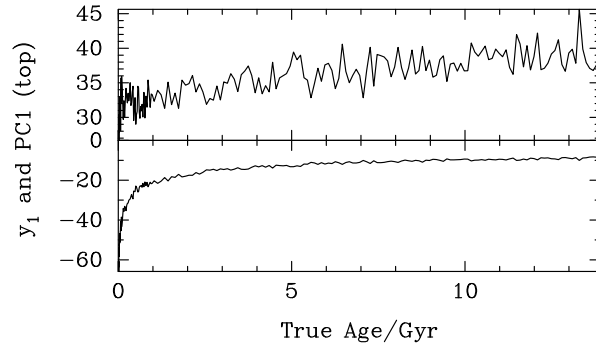


FIG. 11. First principal component (PC1) and y_1 versus age. One in every 10 models was used to do the PCA. In the presence of noise, at a level of $S/N < 2$ per bin, y_1 is almost monotonic with age, whereas PC1, although correlated with age, is not a good age estimator.